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CHARACTERISTIC SURFACES IN GAS FLOWS†

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It is shown that weak discontinuities of three types may occur in flows of an inviscid heat-conducting gas: on sonic lines, on contact surfaces and on a thermal wave front. Weak discontinuities may occur in flows of a viscous non-heat-conducting gas, but only of one type – contact weak discontinuities. © 2002 Elsevier Science Ltd. All rights reserved.

It is well know that the system of gas dynamic equations (for an inviscid non-heat-conducting compressible continuous medium) is of hyperbolic type and therefore, in particular, admits of gas flows with weak discontinuities on sonic lines or contact characteristics [1, 2]. This property enables one to solve complex and important problems (see, e.g. [3]). The complete system of Navier–Stokes equations [4], which describes viscous heat-conducting gas flows, is of mixed type and also admits of flows with weak discontinuites, either on a thermal wave front [5] or on a contact surface [6]. The investigation of flows, both of an inviscid heat-conducting gas [7, 8] and of a viscous non-heat-conducting gas [9] is of interest in relation to the problem of developing high energy densities [10].

To describe thermodynamically perfect gas flows whose equations of state are

$$p = R\rho T,\ e = c_{\upsilon 0}T;\ \ R,\ c_{\upsilon 0} = \text{const} > 0 \tag{1}$$

(p is the pressure, ρ is the density, T is the temperature and e is the internal energy), one can take ρ and T as the independent thermodynamic variables. All the other thermodynamic parameters may then be expressed in terms of these two via the fundamental thermodynamic identity $TdS = de + pd\,(1/\rho)$, where S is the entropy. In particular, the square of the speed of sound is $c^2 = (\partial p/\partial\rho)|_{s=\text{const}} = R\gamma/T$, where $\gamma = 1 + R/c_{\upsilon 0} > 1$ is the adiabatic exponent of the gas. For gas flows with equations of state (1) we shall henceforth consider the complete system of Navier–Stokes equations in the form introduced in [11].

We will first investigate the case of an inviscid heat-conducting gas: the coefficients of dynamic and volume viscosity μ and μ' are set equal to zero. Then the system of equations for the flow is

$$\begin{aligned}
&\frac{\partial\rho}{\partial t} + \mathbf{u}\cdot\nabla\rho + \rho\,\mathrm{div}\,\mathbf{u} = 0\\
&\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{1}{\gamma}\left(\frac{T}{\rho}\nabla\rho + \nabla T\right) = 0\\
&\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T + (\gamma-1)T\,\mathrm{div}\,\mathbf{u} = \frac{1}{\rho}(\varkappa\Delta T + \nabla\varkappa\cdot\nabla T)
\end{aligned} \tag{2}$$

where t is the time, $\mathbf{u} = \{u_1, u_2, u_3\}$ is the velocity vector of the gas and $\varkappa = \varkappa\,(\rho, T)$ is the thermal conductivity.

This system may be written in the standard way in terms of dimensionless variables by introducing suitable positive constants L_*, ρ_* and T_*; the unit of velocity is taken to be the speed of sound $u_* = c_* = \sqrt{R\gamma T_*}$, so that the dimensionless speed of sound is $c = \sqrt{T}$.

We will now determine when a surface C in the space of the variables t, x_j ($j = 1, 2, 3$) is a characteristic surface of system (1). To do this we will assume that the surface C is specified in the form

$$x_1 = \psi(t, x_2, x_3)$$

where the function $\psi(t, x_2, x_3)$ is assumed to have finite first-order derivatives.

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We make the change of variables

$$\tau = t,\ \ \theta = \varphi(t,x_1,x_2,x_3),\ \ \xi_i = x_i,\ \ i=2,3$$

$$\varphi(t,x_1,x_2,x_3) = x_1 - \psi(t,x_2,x_3)$$

The Jacobian of this transformation is equal to unity. Under this change of variables, the surface C: $\varphi(t,x_1,x_2,x_3) = 0$ becomes the new coordinate plane $\theta = 0$. Using the formulae for the transformation of derivatives

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + \frac{\partial \varphi}{\partial t}\frac{\partial}{\partial \theta},\ \ \frac{\partial}{\partial x_1} = \frac{\partial}{\partial \theta},\ \ \frac{\partial}{\partial x_i} = \frac{\partial}{\partial \xi_i} + \frac{\partial \varphi}{\partial x_i}\frac{\partial}{\partial \theta},\ \ i=2,3$$

we can write system (2) in the form

$$\frac{\partial \rho}{\partial \tau} + (D\varphi)\frac{\partial \rho}{\partial \theta} + \sum_{i=2}^{3} u_i \frac{\partial \rho}{\partial \xi_i} + \rho\left(\sum_{j=1}^{3}\frac{\partial \varphi}{\partial x_j}\frac{\partial u_j}{\partial \theta} + \sum_{i=2}^{3}\frac{\partial u_i}{\partial \xi_i}\right) = 0$$

$$\frac{\partial u_1}{\partial \tau} + (D\varphi)\frac{\partial u_1}{\partial \theta} + \sum_{i=2}^{3} u_i \frac{\partial u_1}{\partial \xi_i} + \frac{1}{\gamma}\left(\frac{T}{\rho}\frac{\partial \rho}{\partial \theta} + \frac{\partial T}{\partial \theta}\right) = 0$$

$$\frac{\partial u_l}{\partial \tau} + (D\varphi)\frac{\partial u_l}{\partial \theta} + \sum_{i=2}^{3} u_i \frac{\partial u_l}{\partial \xi_i} + \frac{1}{\gamma}\left(\frac{T}{\rho}\frac{\partial \varphi}{\partial x_l}\frac{\partial \rho}{\partial \theta} + \frac{T}{\rho}\frac{\partial \rho}{\partial \xi_l} + \frac{\partial \varphi}{\partial x_l}\frac{\partial T}{\partial \theta} + \frac{\partial T}{\partial \xi_l}\right) = 0,\ \ l=2,3$$

$$\frac{\partial T}{\partial \tau} + (D\varphi)\frac{\partial T}{\partial \theta} + \sum_{i=2}^{3} u_i \frac{\partial T}{\partial \xi_i} + (\gamma-1)T\left(\sum_{j=1}^{3}\frac{\partial \varphi}{\partial \xi_j}\frac{\partial u_j}{\partial \theta} + \sum_{i=2}^{3}\frac{\partial u_i}{\partial \xi_i}\right) = \tag{3}$$

$$= \frac{1}{\rho}\Bigg\{\varkappa\left[\sum_{j=1}^{3}\left(\frac{\partial \varphi}{\partial x_j}\right)^2\frac{\partial^2 T}{\partial \theta^2} + \sum_{i=2}^{3}\left(2\frac{\partial \varphi}{\partial x_i}\frac{\partial^2 T}{\partial \theta \partial \xi_i} + \frac{\partial^2 T}{\partial \xi_i^2}\right)\right] +$$

$$\frac{\partial \varkappa}{\partial \rho}\left[\frac{\partial \rho}{\partial \theta}\frac{\partial T}{\partial \theta} + \sum_{i=2}^{3}\left(\frac{\partial \varphi}{\partial x_i}\frac{\partial \rho}{\partial \theta} + \frac{\partial \rho}{\partial \xi_i}\right)\left(\frac{\partial \varphi}{\partial x_i}\frac{\partial T}{\partial \theta} + \frac{\partial T}{\partial \xi_i}\right)\right] +$$

$$+ \frac{\partial \varkappa}{\partial T}\left[\left(\frac{\partial T}{\partial \theta}\right)^2 + \sum_{i=2}^{3}\left(\frac{\partial \varphi}{\partial x_i}\frac{\partial T}{\partial \theta} + \frac{\partial T}{\partial \xi_i}\right)^2\right]\Bigg\}$$

where

$$D\varphi = \frac{\partial \varphi}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial \varphi}{\partial x_j};\ \ \ \frac{\partial \varphi}{\partial t} = -\psi_\tau(\tau,\xi_1,\xi_2),\ \ \frac{\partial \varphi}{\partial x_1} = 1,\ \ \frac{\partial \varphi}{\partial x_i} = -\psi_{\xi_i}(\tau,\xi_1,\xi_2),\ \ i=2,3$$

We introduce the notation

$$\frac{\partial T}{\partial \theta} = T^1,\ \ \frac{\partial^2 T}{\partial \theta^2} = \frac{\partial T^1}{\partial \theta},\ \ \frac{\partial T}{\partial \xi_i} = T^i,\ \ \frac{\partial^2 T}{\partial \theta \partial \xi_i} = \frac{\partial T^1}{\partial \xi_i},\ \ \frac{\partial^2 T}{\partial \xi_i^2} = \frac{\partial T^i}{\partial \xi_i},\ \ i=2,3$$

in system (3) and add three more equations for the three new unknown functions T^j ($j = 1, 2, 3$):

$$\frac{\partial T}{\partial \theta} = T^1,\ \ \frac{\partial T^i}{\partial \theta} = \frac{\partial T^1}{\partial \xi_i},\ \ i=2,3$$

This yields a quasi-linear system of equations (too cumbersome to be given here) involving only the first-order derivatives of the vector of unknown functions $\mathbf{U}$ with eight components ρ, u_j, T, T^j ($j = 1, 2, 3$). The determinant of the matrix A_0, which is the coefficient of the vector $\partial \mathbf{U}/\partial \theta$ (also too cumbersome to be given here), is not only easily evaluated but also factorized:

$$\det A_0 = \frac{1}{\rho}\Phi \varkappa (D\varphi)^2 \left[(D\varphi)^2 - \frac{1}{\gamma} T\Phi \right], \quad \Phi = \sum_{j=1}^{3} \left(\frac{\partial \varphi}{\partial x_j} \right)^2$$

Thus, characteristics can only exist in an inviscid thermally conducting gas flow if one of the three factors vanishes, that is, one of the following three equalities must hold

$$\varkappa = 0, \quad D\varphi = 0, \quad (D\varphi)^2 - \frac{1}{\gamma} T\Phi = 0$$

If the flow satisfies the conditions

$$\varkappa(\rho, T)|_{T=0} = 0, \quad \varkappa(\rho, T)|_{T>0} > 0$$

it may be treated as a thermal wave propagating on a cold background.

Equating the second factor to zero gives a contact surface C°.

Equating the third factor to zero gives two differential equations, which differ in only one sign and can therefore be written uniformly:

$$\frac{\partial \varphi}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial \varphi}{\partial x_j} \pm \frac{1}{\sqrt{\gamma}} \sqrt{T\Phi} = 0$$

Thus, in unsteady inviscid thermally conducting gas flows of any dimension there are always two characteristics, henceforth denoted by $C^{\pm}_{\varkappa}$, also known as sonic lines. The propagation velocity of these characteristics (relative to the flow under consideration) in a heat-conducting gas is

$$c_{\varkappa} = \sqrt{T} / \sqrt{\gamma} = c / \sqrt{\gamma} \tag{4}$$

and is independent of the thermal conductivity. Here c is the propagation velocity of the sonic lines of an inviscid non-heat-conducting gas. Consequently, the speed of sound in a heat-conducting gas is strictly less than in a non-heat-conducting gas.

This explains the effect, observed in computations [12], that a compression wave front in a heat-conducting gas "lags behind" the analogous compression wave front in a non-heat-conducting gas.

Remark. Naturally, the above considerations are also possible in the case of other equations of state $p = p(\rho, T)$. Then the velocity at which the sonic lines propagate in heat-conducting gas flows will be given by the formula $c_{\varkappa}(\rho, T) = \sqrt{\partial p(\rho, T)/\partial \rho}$. In gas dynamics one sometimes considers (as, e.g., in [13]), the case of isothermal flows, when it is assumed that, because of infinite thermal conductivity, the gas temperature instantaneously becomes equal throughout the volume under consideration, and in such flows, therefore, $T = \text{const}$. In that case the energy equation is omitted, while the remaining equations (continuity and conservation of momentum) constitute a hyperbolic system in which the propagation velocity of sonic lines ("the isothermal speed of sound") is $(\sqrt{\partial p(\rho, T)/\partial \rho})|_{T=\text{const}}$. In this paper, unlike the isothermal case, we consider the general situation with $T \neq \text{const}$ and do not omit the energy equation; the thermal conductivity is assumed to be finite and strictly positive. Naturally, the "isothermal speed of sound" is identical with the speed of sound $c_{\varkappa}(\rho, T)$ in flows with finite thermal conductivity if one puts $T = \text{const}$ in $c_{\varkappa}(\rho, T)$. In [14], on the assumption that the thermal conductivity is large but finite, a particular solution of the linearized system (3) was constructed in the form of a travelling wave; it was then shown by asymptotic analysis of the dispersion relation that in the low-frequency case such a wave travels at the "adiabatic speed of sound" $(\sqrt{\partial p(\rho, S)/\partial \rho})|_{S=\text{const}}$, while in the high-frequency case one obtains the "isothermal speed of sound" $(\sqrt{\partial p(\rho, T)/\partial \rho})|_{T=\text{const}}$. In contrast to this result, it was shown above that, for any positive value of the thermal conductivity, system (3) (unlinearized) has sonic lines propagating at a velocity $c_{\varkappa}(\rho, T)$, irrespective of the form of the selected solution of system (3). We also observe that the system obtained by linearizing system (3), relative to the solution describing uniform rest at constant parameters $\rho = \rho_{00}$, $T = T_{00}$, also has sonic lines, whose propagation velocity is naturally constant and equal to $c_{\varkappa}(\rho, T)|_{\rho=\rho_{00}, T=T_{00}}$.

In our further analysis of heat-conducting gas flows in the neighbourhood of the characteristics $C^{\pm}_{\varkappa}$, in order to avoid complications, we shall consider the case of uniform plane-symmetric flows: $\partial/\partial x_2 = \partial/\partial x_3 = u_2 = u_3 = 0$, change the notation to $x_1 = x$, $u_1 = u$ and change variables:

$$\tau = t,\ \theta = x - \psi(t)$$

To simplify the notation, we will retain the previous notation for the variable t. System (3) may then be written in the form

$$\begin{aligned}
&\rho_t + (u - \psi')\rho_\theta + \rho u_\theta = 0 \\
&u_t + (u - \psi')u_\theta + \frac{1}{\gamma}\left(\frac{T}{\rho}\rho_\theta + T_\theta\right) = 0 \\
&T_t + (u - \psi')T_\theta + (\gamma - 1)Tu_\theta = \frac{1}{\rho}(\varkappa T_{\theta\theta} + \varkappa_\rho \rho_\theta T_\theta + \varkappa_T T_\theta^2)
\end{aligned} \tag{5}$$

In order for the axis $\theta = 0$ to be one of the characteristics $C_\varkappa^\pm$, the function $x = \psi(t)$ must satisfy one of the two differential equations

$$\psi' = u \pm \sqrt{T/\gamma}$$

Derivatives issuing from the characteristics $C_\varkappa^\pm$ will be denoted by

$$f_k(t) = \partial^k f(t,\theta)/\partial\theta^k\,|_{\theta=0},\quad k = 0,1,2,\ldots;\quad f = \rho, u, T$$

Let us suppose that one of the characteristics is defined in terms of a function $\psi(t)$: $C_\varkappa^\pm : x = \psi(t)$, and that the values of all the unknown functions and (because of the form of system (5)) the thermal flow $T_x|_{C_\varkappa^\pm} = T_\theta|_{\theta=0}$ on that characteristic are also given:

$$\begin{aligned}
&\rho\,|_{\theta=0} = \rho_0(t) > 0,\quad u\,|_{\theta=0} = u_0(t) \\
&T\,|_{\theta=0} = T_0(t) > 0,\quad T_\theta\,|_{\theta=0} = T_1(t)
\end{aligned} \tag{6}$$

Then

$$u_0 - \psi' = \mp\sqrt{T_0/\gamma} \neq 0 \tag{7}$$

If we put $\theta = 0$ in system (5) and takes relations (6) and (7) into consideration, one obtains the following results. The first equation is equivalent to the relation

$$\rho_1 = \pm(\rho_0 u_1 + \rho_0')/\sqrt{T_0/\gamma} \tag{8}$$

A linear combination of the first two equations (the first equation is multiplied by $\pm\sqrt{T_0/\gamma}/\rho_0$ and the product added to the second) yields

$$T_1 = \mp\sqrt{\gamma T_0}\,\rho_0'/\rho_0 - \gamma u_0' \tag{9}$$

Equation (9) is an additional restriction, superimposed on condition (6); it is a necessary condition for the problem with data on a characteristic to be solvable [3, 15]. In our case the characteristic in question is $C_\varkappa^\pm$. The third equation of system (5) with $\theta = 0$ uniquely defines the second derivative of the temperature with respect to θ:

$$T_2 = \frac{\rho_0}{\varkappa_0}\left[T_0' \mp \sqrt{\frac{T_0}{\gamma}}T_1 + (\gamma - 1)T_0 u_1\right] - \frac{1}{\varkappa_0}(\varkappa_{\rho 0}\rho_1 T_1 + \varkappa_{T0} T_1^2)$$

In this and the following formulae, the subscript 0 on the thermal conductivity and its partial derivatives with respect to ρ and T means that these functions are considered at $\rho = \rho_0(t)$, $T = T_0(t)$.

If we differentiate the first two equations of system (5) with respect to θ, put $\theta = 0$ and take the form of f_0 into consideration, then the same linear combination of these equations yields the transport equation for $u_1(t)$:

$$u_1' + u_1^2 + A(t)u_1 + B(t) = 0 \tag{10}$$

The functions $A(t)$ and $B(t)$ are expressed in terms of $f_0(t)$ and are too cumbersome to be reproduced here.

Equation (10) is the general Riccati equation, which is not integrable in quadratures for arbitrary $A(t)$ and $B(t)$. Since Eq. (10) is non-linear, some of its particular solutions become infinite at finite values of t. This property of the solutions of the transport equations is known in gas dynamics as the gradient catastrophe (see e.g., [16]).

If constant values of the unknown functions are given on the characteristic $C_\varkappa^\pm$

$$\rho_0(t) = \rho_{00} = \text{const} > 0,\quad u_0(t) = 0,\quad T_0(t) = T_{00} = \text{const} > 0 \tag{11}$$

then, first, the following values are also uniquely defined on $C_\varkappa^\pm$

$$T_1(t) = 0,\quad T_2(t) = \alpha u_1(t),\quad \alpha = (\gamma - 1)\rho_{00}T_{00}/\varkappa_0 > 0 \tag{12}$$

and, second, in Eq. (10),

$$A(t) = \beta,\ \beta = \alpha/(2\gamma) > 0;\ B(t) = 0$$

In that case Eq. (10) has a particular solution

$$u_1(t) = \frac{\beta u_{10}}{(u_{10} + \beta)e^{\beta t} - u_{10}},\quad u_1(t)|_{t=0} = u_{10} \tag{13}$$

And if, for example, $u_{10} < 0$, $u_{10} + \beta < 0$, then the onset of the gradient catastrophe occurs at a time

$$t_* = \frac{1}{\beta}\ln\left(\frac{u_{10}}{u_{10} + \beta}\right) > 0$$

The coefficient of u_1 in (8), as well as that of its derivative in (10), do not vanish. This makes it possible, first, to write a transport equation for ρ_1 as well, and, second, to establish the existence of flows in the neighbourhood of the characteristics $C_\varkappa^\pm$.

Theorem 1. Suppose initial data (6) satisfying conditions (7) and (9) are given on a characteristic $C_\varkappa^\pm$ of system (5) and one of the following conditions holds: either

$$\rho(t,\theta)|_{t=0} = \rho^0(\theta),\quad \rho^0(\theta)|_{\theta=0} = \rho_0(t)|_{t=0} \tag{14}$$

or

$$u(t,\theta)|_{t=0} = u^0(\theta),\quad u^0(\theta)|_{\theta=0} = u_0(t)|_{t=0} \tag{15}$$

Then, if all input data are analytic in a neighbourhood of the point ($t = 0$, $\theta = 0$), problems (5), (6), (14) and (5), (6), (15) have unique analytic solutions.

The gist of the problems formulated in this theorem is to derive given distributions at time $t = 0$ of either the density $\rho = \rho^0$ or the velocity $u = u^0$ of the gas, which continuously approximate the background flow across a weak discontinuity $C_\varkappa^\pm$. The proof of Theorem 1 will not be given here, since it essentially reduces these problems to a characteristic Cauchy problem of standard form [3, 15], and in the main duplicates the proof of the analogous facts for the system of equations of gas dynamics [17].

It also follows from relation (7) that when $T > 0$ the characteristic $C_\varkappa^\pm$ and trajectory of motion of the particle (which may be understood as the trajectory of motion of an impermeable piston) never touch one another. Consequently, one can formulate the problem of the smooth motion of an impermeable piston.

Theorem 2. Suppose initial data (6) satisfying conditions (7) and (9) are given on a characteristic $C_\varkappa^\pm$ of system (5) and the following condition holds

$$u(t,\theta)|_{\theta = x_p(t) - \psi(t)} = x_p'(t),\quad x_p(0) = \psi(0),\quad u_0(0) = x_p'(0) \tag{16}$$

Then if all the input data are analytic in the neighbourhood of the point ($t = 0, \theta = 0$), then a unique analytic solution of problem (5), (6), (16) exists.

Condition (16) stipulates that a piston which begins to move at time $t = 0$ from the point $x = \psi(0)$, moving smoothly in the gas according to a law $x = x_p(t)$, does not leak: the piston velocity at time $t = 0$ equals the gas velocity at the piston. The proof of this theorem will also be omitted, since it largely duplicates that of the corresponding theorem for the system of equations of gas dynamics [18].

Theorems 1 and 2 can be extended to the case of three-dimensional unsteady flows.

Since the solution of the problem of a smoothly moving piston is unique, it follows that the conditions formulated in Theorem 2 uniquely define the entire gas flow from the piston to the characteristic $C_\varkappa^\pm$, so that the temperature and heat flux at the piston itself are uniquely defined. It is by no means certain that these values will be identical with any prescribed ones, e.g., a constant temperature or zero heat flux at the piston. This is illustrated by the following argument.

Suppose we are given a uniform heat-conducting gas at rest, with parameters (11), and a sonic line therein, say $C_\varkappa^\pm$, propagating from left to right: $x = \sqrt{T_{00}/\gamma}\,t + \psi(0)$, produced when an impermeable piston moves smoothly ($x_0'(0) = 0, x_p''(0) > 0$) into the gas. Such a piston creates a compression wave: $u_x|_{C_\varkappa^\pm} < 0$, $\rho_x|_{C_\varkappa^\pm} < 0$. But it then follows from (12) that $T_2 < 0$, and at small t the gas temperature throughout the region of the compression wave, including the piston, will be less than the temperature of the uniform background through which the characteristic $C_\varkappa^\pm$ is propagating. Thus, simultaneously with the compression of the gas, heat will also escape from the gas through the piston. But if the compressing piston is thermally insulated, or if heat is supplied through it to the gas, then of necessity one will have the inequality $T_x|_{C_\varkappa^\pm} > 0$. However, by (9), conditions (11) will then fail to hold on $C_\varkappa^\pm$ – the background flow will no longer be uniform and at rest. This conclusion also follows from general physical considerations: the propagation of heat will overtake the characteristic $C_\varkappa^\pm$, which is moving at a finite velocity (4), it will modify the gas parameters ahead of the characteristic and will thus influence its propagation velocity. At the same time, computations of compression waves in a heat-conducting gas show [12] that the front of a weak discontinuity is nevertheless preserved, as follows from the presence of $C_\varkappa^\pm$-characteristics in heat-conducting gas flows.

The formulae and arguments presented here constitute concrete mathematical confirmation of the general conclusion that solutions of system (2) take into account both mechanisms by which disturbances are transmitted: by means of elastic interaction and by means of thermal conductivity, for each of which phenomena there is a specific velocity (finite and infinite, respectively) at which the disturbances propagate.

Returning to system (2), let us consider the contact surface C° defined by the equation

$$\frac{\partial\varphi}{\partial t} + \sum_{i=1}^{3} u_i \frac{\partial\varphi}{\partial x_i} = 0$$

As shown previously, a contact surface in inviscid heat-conducting gas flows is a characteristic of multiplicity two. As the dimensionality of the flow – the sum of the dimension of the space of independent variables and the number of unknown functions – is reduced, the multiplicity of the characteristic also decreases: if $\partial/\partial x_3 = u_3 = 0$, the multiplicity is one; if $\partial/\partial x_2 = \partial/\partial x_3 = u_2 = u_3 = 0$, a contact surface in inviscid heat-conducting gas flows is not a characteristic. As is well known, in the case of the system of equations of gas dynamics, the contact surface is a characteristic of multiplicity three; as the dimensionality of the problem decreases this multiplicity also decreases, and in the case of unsteady potential isentropic flows the surface is no longer a characteristic [1]. If one reduces only the dimension of the space of independent variables but not the number of unknown functions (for example, if $\partial/\partial x_3 = 0$ but $u_3 \neq 0$), the multiplicity of the contact surface does not change.

To avoid unnecessary complication, the gas flow in the neighbourhood of a contact surface is considered in the case when $\partial/\partial x_3 = u_3 = 0$. One then uses the notation $u_1 = u, u_2 = v, x_1 = x, x_2 = y$, system (3) is somewhat simplified and rewritten in an appropriate form. As before, we introduce the notation f_k for the values of the k-th derivatives with respect to θ at $\theta = 0$.

Suppose we are given a function $\psi(t, y)$ defining a characteristic C°: $x = \psi(t, y)$; suppose we are also given the values of all the unknown functions and the heat flux $T_x|_{C^\circ} = T_\theta|_{\theta=0}$ on the characteristic:

$$\rho|_{\theta=0} = \rho_0(\tau,\xi) > 0, \quad u|_{\theta=0} = u_0(\tau,\xi), \quad v|_{\theta=0} = v_0(\tau,\xi) \qquad (17)$$

$$T|_{\theta=0} = T_0(\tau,\xi) > 0, \quad T_\theta|_{\theta=0} = T_1(\tau,\xi)$$

Then, first, the C°-characteristic itself is given by the equation

$$u_0 - \psi_\tau - \psi_\xi v_0 = 0$$

Second, the first equation of system (5) at $\theta = 0$ yields the equation

$$u_1 = \psi_\xi v_1 + C(\tau,\xi),\quad C(\tau,\xi) = -\frac{\rho_{0\tau} + v_0\rho_{0\xi}}{\rho_0} + v_{0\xi}$$

Third, a linear combination of the second and third equations at $\theta = 0$ leads to the necessary condition for a Cauchy problem with data on the characteristic C° to be solvable

$$\psi_\xi u_{0\tau} + v_{0\tau} + v_0(\psi_\xi u_{0\xi} + v_{0\xi}) + \frac{1}{\gamma}\left(\frac{T_0\rho_{0\xi}}{\rho_0} + T_{0\xi}\right) = 0$$

This condition is an additional restriction, superimposed on initial data (17), which (unlike condition (9) for $C^{\pm}_{\varkappa}$) does not involve the heat flux on C°.

The transport equation – the equation for the derivative $v_1(\tau, \xi)$ issuing from the surface C° – is obtained from the appropriate linear combination of the second and third equations, after first differentiating them with respect to θ and considering them at $\theta = 0$

$$v_{1\tau} + v_0(\tau,\xi)v_{1\xi} + D(\tau,\xi)v_1 + E(\tau,\xi) = 0$$

where

$$D(\tau,\xi) = C(\tau,\xi) + \frac{\psi_\xi\psi_{\xi\xi}v_0 + \psi_\xi u_{0\xi} + v_{0\xi}}{1+\psi_\xi^2}$$

(the function $E(\tau, \xi)$ is too cumbersome to be reproduced here). Since the transport equation is a first-order linear partial differential equation, the singularities of its solutions are known [19].

Finally, the last possibility for a weak discontinuity to exist in inviscid heat-conducting gas flows: continuous adherence of the cold background (in which $T = \varkappa = 0$) to a thermal wave (in which $T > 0$ and $\varkappa > 0$). For simplicity, this situation will be considered in the case of two-dimensional symmetric flows satisfying system (5).

Theorem 3. Suppose the thermal conductivity $\varkappa(\rho, T)$ satisfies the conditions

$$\varkappa(\rho,T)|_{T=0} = 0,\quad \varkappa_\rho(\rho,T)|_{T=0} = 0,\quad \varkappa_T(\rho,T)|_{T=0} > 0 \tag{18}$$

and system (5) has a solution

$$\rho = \rho_{00} = \text{const} > 0,\quad u \equiv 0,\quad T \equiv 0 \tag{19}$$

henceforth referred to as the cold uniform background. If the given function $x = \psi(t)$ is such that

$$\psi'(0) \neq 0 \tag{20}$$

then, provided all the input data are analytic in some neighbourhood of the point $t = 0, x = \psi(0)$, the problem formulated above has another analytic solution – besides (19) – which continuously approximates it on the line $\theta = 0$, that is, for $x = \psi(t)$.

This second solution is constructed as an infinite series

$$f(t,\theta) = \sum_{k=0}^{\infty} f_k(t)\frac{\theta^k}{k!},\quad f = \rho, u, T \tag{21}$$

where the quantities ρ_0, u_0 and T_0 are taken equal to their respective values for the cold uniform background, i.e.

$$\rho_0 = \rho_{00},\ u_0 = 0,\ T_0 = 0$$

We put $\theta = 0$ in system (5) and take into consideration the form of the zeroth coefficients and condition (20). As a result we obtain three relations

$$\rho_1 = \frac{\rho_{00} u_1}{\psi'}, \quad u_1 = \frac{T_1}{\gamma \psi'}, \quad -\psi' T_1 = \frac{\varkappa_{T0}}{\rho_{00}} T_1^2$$

The last equality is possible in two cases: $T_1 = 0$ or

$$T_1 = -\psi' \rho_{00} / \varkappa_{T0} \tag{22}$$

The case $T_1 = 0$ leads to solution (19), so that henceforth we will consider only the non-zero value of (22) for T_1. This value uniquely defines u_1 and ρ_1, which are also not zero.

We now differentiate each equation of system (5) k times with respect to θ, set $\theta = 0$ and take the values of f_1 into consideration. This yields relations

$$-\psi' \rho_{k+1} + \rho_{00} u_{k+1} = F_k, \quad -\psi' u_{k+1} + \frac{1}{\gamma} T_{k+1} = G_k$$

$$\left[-(k+2) \frac{\psi'}{\varkappa_{T0}} T_1 + \psi' \rho_{00} \right] T_{k+1} = H_k$$

where F_k, G_k and H_k depend on f_l $(l = 0, 1, \dots, k)$. By (22), the bracketed expression in the last equation is equal to $-(k+1)\,\psi' \rho_{00}$. Hence these relations uniquely define f_{k+1}, so that series (21) has been formally constructed. To prove that the series converges, we construct a majorant problem with an analytic solution. This construction is identical with that of the analogous majorant problem in the case of a viscous heat-conducting gas flow [5] and will not be reproduced here.

The analytic solution (21) thus obtained is a thermal wave (TW) propagating through the cold uniform background (19) and continuously coinciding with it at the TW front, whose trajectory of motion is given by the function

$$x = \psi(t) \tag{23}$$

It follows from (22) that the sign of T_1 is the reverse of the sign of ψ'. If $\psi' > 0$, the TW front moves to the right through the cold background. Since then $T_1 < 0$, the TW itself (in which $T > 0$) lies to the left of the front (23) and the cold background to its right. if $\psi' < 0$, the TW front propagates to the left through the cold background, $T_1 > 0$, the TW, in which $T > 0$, lies to the right of the curve (23), and the cold background to its left. It follows from the formulae for u_1 and ρ_1 that the TW is a compression wave: behind the wave front there is an increase not only in temperature but also in the density of the gas and the magnitude of its velocity.

Theorem 3 admits of a natural extension to the case of a three-dimensional TW propagating in any cold flow – a solution of system (2) with $T \equiv 0$. A TW also exists when $\varkappa = \varkappa_0 \sqrt{T}$, $\varkappa_0 = \text{const} > 0$.

In conclusion, we will briefly consider the case of a viscous non-heat-conducting gas. To that end, we equate the thermal conductivity to zero, $\varkappa = 0$, in the complete system of Navier–Stokes equations [11]; for simplicity, we will assume that the viscosity coefficients are constant, the first being $\mu = \mu_0 = \text{const} > 0$ and the second $\mu' = 0$. To further simplify the arguments, we will consider two-dimensional symmetric flows and take some curve (23) as the new coordinate axis $\theta = 0$. As a result, we obtain the system of equations

$$\begin{aligned} &\rho_t + (u - \psi')\rho_\theta + \rho u_\theta = 0 \\ &u_t + (u - \psi')u_\theta + \frac{1}{\gamma}\left(\frac{T}{\rho}\rho_\theta + T_\theta \right) = \mu_0 \frac{1}{\rho} u_{\theta\theta} \\ &T_t + (u - \psi')T_\theta + (\gamma - 1) T u_\theta = \gamma(\gamma - 1)\mu_0 \frac{1}{\rho} u_\theta^2 \end{aligned} \tag{24}$$

If (23) is a contact curve, i.e., the trajectory of motion of some gas particle satisfies the equation $\psi'(t) = u(t, \psi(t))$, then it follows directly from the form of system (24) that the $\theta = 0$ axis is a characteristic of multiplicity two. Under these conditions the determinant of the matrix which is the coefficient of the vector of derivatives issuing from the $\theta = 0$ axis is equal to $-\mu_0(u - \psi')^2/\rho$.

The initial data on the curve $\theta = 0$ for system (24) have the form

$$\rho\,|_{\theta=0} = \rho_0(t) > 0,\quad u\,|_{\theta=0} = u_0(t)$$
$$u_\theta\,|_{\theta=0} = u_1(t),\quad T\,|_{\theta=0} = T_0(t) > 0 \tag{25}$$

where

$$u_0(t) = \psi'(t)$$

If we put $\theta = 0$ in system (24) and take initial conditions (25) into consideration, we obtain three equations:

$$\rho_0' + \rho_0 u_1 = 0$$
$$u_0' + \frac{1}{\gamma}\left(\frac{T_0}{\rho_0}\rho_1 + T_1\right) = \mu_0 \frac{1}{\rho_0} u_2 \tag{26}$$
$$T_0' + (\gamma - 1)T_0 u_1 = \gamma(\gamma - 1)\mu_0 \frac{1}{\rho_0} u_1^2$$

The first and third equations are two necessary conditions for the problem to be solvable – two additional restrictions superimposed on the initial data, since the characteristic in question has multiplicity two [3, 15]. The second equation uniquely defines u_2 in terms of initial data (25) and a linear combination of ρ_1, T_1.

If the first and third equations of (24) are differentiated with respect to θ and we put $\theta = 0$ in them, then, taking relations (25) and (26) into consideration, we obtain a linear system of transport equations

$$\rho_1' + a_{11}(t)\rho_1 + a_{12}(t)T_1 = b_1(t)$$
$$T_1' + a_{21}(t)\rho_1 + a_{22}(t)T_1 = b_2(t)$$

The functions $a_{ij}(t)$, $b_i(t)$ $(t, j = 1, 2)$ are determined by the initial data, but are too cumbersome to reproduce here. As is well known, the solutions of a linear system of ordinary differential equations may have singularities only at t values where the coefficients and right-hand sides of the equations have singularities.

If the density is constant on the contact curve (23), that is

$$\rho_0(t) = \rho_{00} = \text{const} > 0$$

then necessarily

$$u_1(t) = 0,\ T_0(t) = T_{00} = \text{const} > 0$$

Then all the coefficients in the system of transport equations become constant:

$$a_{11} = \frac{\rho_{00}T_{00}}{\gamma\mu_0},\quad a_{12} = \frac{\rho_{00}^2}{\gamma\mu_0}$$
$$a_{21} = \frac{(\gamma - 1)T_{00}^2}{\gamma\mu_0},\quad a_{22} = \frac{(\gamma - 1)\rho_{00}T_{00}}{\gamma\mu_0}$$

In the case when the system of transport equations is homogeneous (for example, if $u_0 = \text{const}$), the functions $\rho_1(t)$ and $T_1(t)$ are appropriate linear combinations of constants and exponential functions with exponent $-Bt$, where $B = \rho_{00}T_{00}/\mu_0$.

Thus, as in viscous heat-conducting gas flows [6], the flow stabilization process (smoothing out of small disturbances) near a contact characteristic in a viscous non-heat-conducting gas is determined by the exponential function $\exp(-Bt)$, where the positive constant B is inversely proportional to μ_0.

If the dimensionality of the problem is increased, the contact surface C^0 remains a characteristic, its multiplicity remains equal to two and there are no other characteristic surfaces in viscous non-heat-conducting gas flows.

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